Explicit Modal Analysis of Axially Loaded Composite Timoshenko Beams Using Symbolic Computation

J. R. Banerjee*
City University,
London, England EC1V OHB, United Kingdom

Introduction

INVESTIGATIONS into the free vibration analysis of composite beams have been well documented.[1–4] A principal finding of this research shows that the effect of material coupling arising from ply orientation on the free vibration behavior of composite beams can be important from the point of view of aeroelastic tailoring.[5] Generally, solutions of vibration problems of composite beams are numerically based because of the huge difficulties involved in seeking closed-form analytical solutions. However, the recent advancement in symbolic computation[6] has made it possible to solve engineering problems by algebra rather than by numerical means. This has motivated investigations in recent years to derive explicit expressions for the frequency equations and mode shapes of simple composite beams[7] and composite Timoshenko beams.[8] The purpose of the present study is to extend these earlier works and derive the frequency equation and mode shape of a cantilever composite Timoshenko beam by taking into account the important effect of an axial load. The proposed theory can be extended to other end conditions of axially loaded composite Timoshenko beams, and it offers prospects for aeroelastic optimization.

Theory

In the usual notation, the governing differential equations in free natural vibration of an axially loaded composite Timoshenko beam shown in Fig. 1 are[4]

$$EI\theta'' + kAG(h' - \theta) + K\psi'' - \rho I\ddot{\theta} = 0 \tag{1}$$

$$kAG(h'' - \theta') - Ph'' - m\ddot{h} = 0 \tag{2}$$

$$GJ\psi'' + K\theta'' - P(I_\alpha/m)\psi'' - I_\alpha\ddot{\psi} = 0 \tag{3}$$

where $h(y, t)$, $\theta(y, t)$, and $\psi(y, t)$ are bending displacement, bending rotation, and twist at a distance y and primes and dots denote differentiation with respect to position y and time t, respectively; ρ is the density of the material; and *EI, GJ, K,* and *kAG* are the bending rigidity, torsional rigidity, bending–torsion coupling rigidity, and shear rigidity of the composite beam respectively. I is the second moment of area of the beam cross section about the X axis; $m = \rho A$ is the mass per unit length, where A is the cross-sectional area. I_α is the polar mass moment of inertia per unit length about the Y axis, θ is the angle of rotation, in radians, of the cross section about the X axis due to bending alone, so that the total slope h' equals the sum of the slopes due to bending and due to shear deformation, and P is a constant compressive axial load passing through the centroid of the cross section. Note that P can be negative so that tension is included. The theory that follows can be applied to composite beams of any general cross section so long as the rigidities *EI, GJ, K,* and kAG are known (either by theory or by experiment), but for the development of the theory, the rectangular cross section is shown in Fig. 1 only for convenience.

If a harmonic variation of h, θ, and ψ, with circular frequency ω, is assumed, then

$$h(y, t) = H(y)e^{i\omega t}, \qquad \theta(y, t) = \Theta(y)e^{i\omega t}$$

$$\psi(y, t) = \Psi(y)e^{i\omega t} \tag{4}$$

where $H(y)$, $\Theta(y)$, and $\Psi(y)$ are the amplitudes of the harmonically varying vertical displacement, bending rotation, and twist, respectively.

Substituting Eqs. (4) into Eqs. (1–3) gives

$$EI\Theta'' + kAG(H' - \Theta) + K\Psi'' + \rho I\omega^2\Theta = 0 \tag{5}$$

$$kAG(H'' - \Theta') - PH'' + m\omega^2 H = 0 \tag{6}$$

$$GJ\Psi'' + K\Theta'' - P(I_\alpha/m)\Psi'' + I_\alpha\omega^2\Psi = 0 \tag{7}$$

When the rules of linear operators and extensive algebraic manipulation are used, Eqs. (5–7) can be combined into one equation, by eliminating all but one of the three variables H, Θ, and Ψ, to give

$$(D^6 + \bar{a}D^4 - \bar{b}D^2 - \bar{c})W = 0 \tag{8}$$

where

$$W = H, \Theta, \quad \text{or} \quad \Psi \tag{9}$$

$$D = \frac{\mathrm{d}}{\mathrm{d}\xi}, \qquad \xi = \frac{y}{L} \tag{10}$$

and

$$\bar{a} = \frac{b^2s^2(b^2c^2 - a^2p^2) + (b^2 - a^2p^2)\{p^2 + b^2r^2(1 - p^2s^2)\} + a^2b^2(1 - p^2s^2)}{(1 - p^2s^2)(b^2c^2 - a^2p^2)}$$

$$\bar{b} = \frac{b^2\{(1 - b^2r^2s^2)(b^2 - 2a^2p^2) - a^2b^2(r^2 + s^2)\}}{(1 - p^2s^2)(b^2c^2 - a^2p^2)}, \qquad \bar{c} = \frac{a^2b^4(1 - b^2r^2s^2)}{(1 - p^2s^2)(b^2c^2 - a^2p^2)} \tag{11}$$

with

$$a^2 = \frac{I_\alpha\omega^2L^2}{GJ}, \qquad b^2 = \frac{m\omega^2L^4}{EI}, \qquad c^2 = 1 - \frac{K^2}{EIGJ}$$

$$r^2 = \frac{I}{AL^2}, \qquad s^2 = \frac{EI}{kAGL^2}, \qquad p^2 = \frac{PL^2}{EI} \tag{12}$$

Note that r^2, s^2, and p^2 describe the effects of rotatory inertia, shear deformation, and axial force, respectively. Any one, two or all three of these parameters can be set to zero so that either the effect of rotatory inertia and/or the effect of shear deformation and/or the effects of axial force can be optionally ignored.

Presented as Paper AIAA2001-1383 at the AIAA/ASME/ASCE/AHS/ASC 42nd Structures, Structural Dynamics and Materials Meeting, Seattle, WA, 16–19 April 2001; received 20 April 2001; revision received 8 April 2002; accepted for publication 10 April 2002.

*Reader, Department of Aeronautical, Civil and Mechanical Engineering, School of Engineering, Northampton Square. Associate Fellow AIAA.

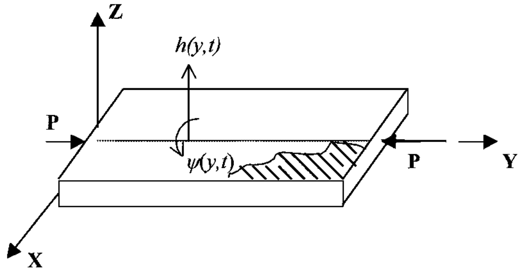

Fig. 1 Coordinate system and notation for an axially loaded composite Timoshenko beam.

The solution of the differential Eq. (8) is[4]

$$W(\xi) = C_1^* \cosh\alpha\xi + C_2^* \sinh\alpha\xi + C_3^* \cos\beta\xi + C_4^* \sin\beta\xi + C_5^* \cos\gamma\xi + C_6^* \sin\gamma\xi \tag{13}$$

where $C_1^* - C_6^*$ are constants and

$$\alpha = \left[2(q/3)^{\frac{1}{2}} \cos(\phi/3) - \bar{a}/3\right]^{\frac{1}{2}}$$

$$\beta = \left[2(q/3)^{\frac{1}{2}} \cos\{(\pi - \phi)/3\} + \bar{a}/3\right]^{\frac{1}{2}}$$

$$\gamma = \left[2(q/3)^{\frac{1}{2}} \cos\{(\pi + \phi)/3\} + \bar{a}/3\right]^{\frac{1}{2}} \tag{14}$$

with

$$q = \bar{b} + \bar{a}^2/3, \qquad \phi = \cos^{-1}\left[(27\bar{c} - 9\bar{a}\bar{b} - 2\bar{a}^3)/\{2(\bar{a}^2 + 3\bar{b})^{\frac{3}{2}}\}\right] \tag{15}$$

Equation (13) represents the solution for the bending displacement $H(\xi)$, bending rotation $\Theta(\xi)$, and torsional rotation $\Psi(\xi)$. Thus,

$$H(\xi) = A_1 \cosh\alpha\xi + A_2 \sinh\alpha\xi + A_3 \cos\beta\xi + A_4 \sin\beta\xi + A_5 \cos\gamma\xi + A_6 \sin\gamma\xi \tag{16}$$

$$\Theta(\xi) = B_1 \sinh\alpha\xi + B_2 \cosh\alpha\xi + B_3 \sin\beta\xi + B_4 \cos\beta\xi + B_5 \sin\gamma\xi + B_6 \cos\gamma\xi \tag{17}$$

$$\Psi(\xi) = C_1 \cosh\alpha\xi + C_2 \sinh\alpha\xi + C_3 \cos\beta\xi + C_4 \sin\beta\xi + C_5 \cos\gamma\xi + C_6 \sin\gamma\xi \tag{18}$$

where A_1–A_6, B_1–B_6, and C_1–C_6 are three different sets of constants.

Substituting Eqs. (16) and (17) into Eq. (6) shows that

$$B_1 = A_1\bar{\alpha}/L, \qquad B_2 = A_2\bar{\alpha}/L, \qquad B_3 = -A_3\bar{\beta}/L$$

$$B_4 = A_4\bar{\beta}/L, \qquad B_5 = -A_5\bar{\gamma}/L, \qquad B_6 = A_6\bar{\gamma}/L \tag{19}$$

where

$$\bar{\alpha} = \{(1 - p^2s^2)\alpha^2 + b^2s^2\}/\alpha, \qquad \bar{\beta} = \{(1 - p^2s^2)\beta^2 - b^2s^2\}/\beta$$

$$\bar{\gamma} = \{(1 - p^2s^2)\gamma^2 - b^2s^2\}/\gamma \tag{20}$$

Then, substituting Eqs. (17) and (18) into Eq. (7) and using Eqs. (19) gives

$$C_1 = A_2k_\alpha/L, \qquad C_2 = A_1k_\alpha/L, \qquad C_3 = A_4k_\beta/L$$

$$C_4 = -A_3k_\beta/L, \qquad C_5 = A_6k_\gamma/L, \qquad C_6 = -A_5k_\gamma/L \tag{21}$$

where

$$k_\alpha = -\frac{\bar{\alpha}\alpha^2k_t}{(1 - a^2p^2/b^2)\alpha^2 + a^2}$$

$$k_\beta = -\frac{\bar{\beta}\beta^2k_t}{(1 - a^2p^2/b^2)\beta^2 - a^2}$$

$$k_\gamma = -\frac{\bar{\gamma}\gamma^2k_t}{(1 - a^2p^2/b^2)\gamma^2 - a^2} \tag{22}$$

with

$$k_t = K/GJ \tag{23}$$

Thus, the bending and twisting rotations given by Eqs. (17) and (18) can be written with the help of Eqs. (19) and (21), in the following form:

$$\Theta(\xi) = [A_1\bar{\alpha} \sinh\alpha\xi + A_2\bar{\alpha} \cosh\alpha - A_3\bar{\beta} \sin\beta\xi + A_4\bar{\beta} \cos\beta\xi - A_5\bar{\gamma} \sin\gamma\xi + A_6\bar{\gamma} \cos\gamma\xi]/L \tag{24}$$

$$\Psi(\xi) = [A_1k_\alpha \sinh\alpha\xi + A_2k_\alpha \cosh\alpha\xi - A_3k_\beta \sin\beta\xi + A_4k_\beta \cos\beta\xi - A_5k_\gamma \sin\gamma\xi + A_6k_\gamma \cos\gamma\xi]/L \tag{25}$$

The expressions for the bending moment $M(\xi)$, shear force $S(\xi)$, and torque $T(\xi)$ can be written as[4]

$$M(\xi) = -W_2[A_1g_\alpha \cosh\alpha\xi + A_2g_\alpha \sinh\alpha\xi - A_3g_\beta \cos\beta\xi - A_4g_\beta \sin\beta\xi - A_5g_\gamma \cos\gamma\xi - A_6g_\gamma \sin\gamma\xi] \tag{26}$$

$$S(\xi) = W_3[A_1f_\alpha \sinh\alpha\xi + A_2f_\alpha \cosh\alpha\xi + A_3f_\beta \sin\beta\xi - A_4f_\beta \cos\beta\xi + A_5f_\gamma \sin\gamma\xi - A_6f_\gamma \cos\gamma\xi] \tag{27}$$

$$T(\xi) = W_1[A_1e_\alpha \cosh\alpha\xi + A_2e_\alpha \sinh\alpha\xi - A_3e_\beta \cos\beta\xi - A_4e_\beta \sin\beta\xi - A_5e_\gamma \cos\gamma\xi - A_6e_\gamma \sin\gamma\xi]/L \tag{28}$$

where

$$W_1 = GJ/L, \qquad W_2 = EI/L^2, \qquad W_3 = EI/L^3 \tag{29}$$

$$g_\alpha = \alpha(\bar{\alpha} + k_bk_\alpha), \qquad g_\beta = \beta(\bar{\beta} + k_bk_\beta)$$

$$g_\gamma = \gamma(\bar{\gamma} + k_bk_\gamma) \tag{30}$$

$$f_\alpha = \alpha\left(g_\alpha + p^2\right) + \bar{\alpha}b^2r^2, \qquad f_\beta = \beta\left(g_\beta - p^2\right) - \bar{\beta}b^2r^2$$

$$f_\gamma = \gamma\left(g_\gamma - p^2\right) - \bar{\gamma}b^2r^2 \tag{31}$$

$$e_\alpha = \alpha\left(k_\alpha + \bar{\alpha}k_t - a^2p^2/b^2\right), \qquad e_\beta = \beta\left(k_\beta + \bar{\beta}k_t - a^2p^2/b^2\right)$$

$$e_\gamma = \gamma\left(k_\gamma + \bar{\gamma}k_t - a^2p^2/b^2\right) \tag{32}$$

with

$$k_b = K/EI \tag{33}$$

Frequency Equation

Now the frequency equation for axially loaded composite Timoshenko beams (Fig. 1) with cantilever boundary conditions can be derived by applying the boundary conditions, and then eliminating the arbitrary constants A_1–A_6. At the built-in end ($\xi = 0$), all displacements are zeros, that is, $H(\xi) = 0$, $\Theta(\xi) = 0$, and $\Psi(\xi) = 0$, whereas at the free end ($\xi = 1$), all forces are zeros, that is, $S(\xi) = 0$, $M(\xi) = 0$, and $T(\xi) = 0$. When these conditions are substituted into Eqs. (16) and (24–28) and the constants A_1–A_6 are eliminated, the frequency equation is first obtained in a 6×6 determinantal form, which is generally solved by numerical means to yield natural frequencies of the composite beam. Here the determinant is expanded algebraically and then simplified very considerably. This formidable task was carried out with the help of the symbolic computation package REDUCE.[9]

The expression for the frequency equation is given in a surprisingly concise form,

$$f(\omega) = \lambda_1 C_\beta C_\gamma C_{h\alpha} + \lambda_2 C_\beta S_\gamma S_{h\alpha} + \lambda_3 C_\gamma S_\beta S_{h\alpha} + \lambda_4 S_\beta S_\gamma C_{h\alpha} + \lambda_5 C_\beta + \lambda_6 C_\gamma + \lambda_7 C_{h\alpha} = 0 \tag{34}$$

where

$$S_{h\alpha} = \sinh\alpha, \qquad S_\beta = \sin\beta, \qquad S_\gamma = \sin\gamma$$
$$C_{h\alpha} = \cosh\alpha, \qquad C_\beta = \cos\beta, \qquad C_\gamma = \cos\gamma \tag{35}$$

$$\lambda_1 = \mu_1\nu_1 f_\gamma + \mu_2\nu_2 f_\alpha + \mu_3\nu_3 f_\beta, \qquad \lambda_2 = \mu_2\nu_1 f_\alpha - \mu_1\nu_2 f_\gamma$$
$$\lambda_3 = \mu_2\nu_3 f_\alpha - \mu_3\nu_2 f_\beta, \qquad \lambda_4 = -\mu_1\nu_3 f_\gamma - \mu_3\nu_1 f_\beta$$
$$\lambda_5 = -\mu_1\nu_2 f_\alpha - \mu_2\nu_1 f_\gamma, \qquad \lambda_6 = -\mu_2\nu_3 f_\beta - \mu_3\nu_2 f_\alpha$$
$$\lambda_7 = \mu_1\nu_3 f_\beta + \mu_3\nu_1 f_\gamma \tag{36}$$

with

$$\mu_1 = e_\alpha g_\beta - e_\beta g_\alpha, \qquad \mu_2 = e_\beta g_\gamma - e_\gamma g_\beta, \qquad \mu_3 = e_\gamma g_\alpha - e_\alpha g_\gamma \tag{37}$$

$$\nu_1 = \bar{\alpha} k_\beta - \bar{\beta} k_\alpha, \qquad \nu_2 = \bar{\beta} k_\gamma - \bar{\gamma} k_\beta, \qquad \nu_3 = \bar{\gamma} k_\alpha - \bar{\alpha} k_\gamma \tag{38}$$

with $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$; k_α, k_β, k_γ; g_α, g_β, g_γ; f_α, f_β, f_γ; e_α, e_β, e_γ; and $C_{h\alpha}$, C_β, C_γ, $S_{h\alpha}$, S_β, S_γ already defined in Eqs. (20), (22), (30), (31), (32), and (35), respectively.

Thus, the natural frequencies of the beam can now be determined from Eq. (34) by computing the value of $f(\omega)$ for a range of frequencies ω and tracking the changes of its sign.

Mode Shapes

Once the natural frequency ω_n is determined from Eq. (34), the corresponding mode shape is obtained in the usual way by fixing one of the six coefficients A_1–A_6 arbitrarily and solving for the remaining five in terms of the arbitrarily chosen one. (This choice is wholly arbitrary, but for the present problem A_2–A_6 are expressed in terms of A_1.) The symbolic computing package REDUCE[9] was used again to derive the mode shapes coefficients in explicit form as follows:

$$A_2 = (\Phi_1/\chi)A_1, \qquad A_3 = (\Phi_2/\chi)A_1, \qquad A_4 = (\Phi_3/\chi)A_1$$
$$A_5 = (\Phi_4/\chi)A_1, \qquad A_6 = (\Phi_5/\chi)A_1 \tag{39}$$

where

$$\Phi_1 = \tau_2(\rho_2 f_\alpha S_{h\alpha} - \rho_1 f_\gamma S_\gamma + \rho_3 f_\beta S_\beta)$$
$$\Phi_2 = \sigma_2 + \xi_3 C_{h\alpha} C_\gamma - \zeta_3 S_{h\alpha} S_\gamma + \delta_3 \tag{40}$$
$$\Phi_3 = \tau_3(\rho_2 f_\alpha S_{h\alpha} - \rho_1 f_\gamma S_\gamma + \rho_3 f_\beta S_\beta)$$
$$\Phi_4 = \sigma_3 + \xi_1 C_{h\alpha} C_\beta + \zeta_1 S_{h\alpha} S_\beta + \delta_1 \tag{41}$$
$$\Phi_5 = \tau_1(\rho_2 f_\alpha S_{h\alpha} - \rho_1 f_\gamma S_\gamma + \rho_3 f_\beta S_\beta)$$
$$\chi = -\zeta_2 C_\beta C_\gamma + \xi_2 S_\beta S_\gamma - \sigma_1 + \delta_2 \tag{42}$$

with

$$\tau_1 = \bar{\alpha} k_\beta - \bar{\beta} k_\alpha, \qquad \tau_2 = \bar{\beta} k_\gamma - \bar{\gamma} k_\beta, \qquad \tau_3 = \bar{\gamma} k_\alpha - \bar{\alpha} k_\gamma \tag{43}$$

$$\rho_1 = g_\alpha C_{h\alpha} + g_\beta C_\beta, \qquad \rho_2 = g_\beta C_\beta - g_\gamma C_\gamma$$
$$\rho_3 = g_\gamma C_\gamma + g_\alpha C_{h\alpha} \tag{44}$$

$$\eta_1 = f_\alpha S_{h\alpha} - f_\beta S_\beta, \qquad \eta_2 = f_\beta S_\beta - f_\gamma S_\gamma$$
$$\eta_3 = f_\gamma S_\gamma - f_\alpha S_{h\alpha} \tag{45}$$

$$\zeta_1 = \tau_3 f_\alpha g_\beta + \tau_2 f_\beta g_\alpha, \qquad \zeta_2 = \tau_3 f_\beta g_\gamma - \tau_1 f_\gamma g_\beta$$
$$\zeta_3 = \tau_2 f_\gamma g_\alpha + \tau_1 f_\alpha g_\gamma \tag{46}$$

$$\xi_1 = \tau_2 f_\alpha g_\beta - \tau_3 f_\beta g_\alpha, \qquad \xi_2 = \tau_1 f_\beta g_\gamma - \tau_3 f_\gamma g_\beta$$
$$\xi_3 = \tau_1 f_\gamma g_\alpha - \tau_2 f_\alpha g_\gamma \tag{47}$$

$$\delta_1 = \tau_2 f_\alpha g_\alpha - \tau_3 f_\beta g_\beta, \qquad \delta_2 = \tau_3 f_\beta g_\beta - \tau_1 f_\gamma g_\gamma$$
$$\delta_3 = \tau_1 f_\gamma g_\gamma - \tau_2 f_\alpha g_\alpha \tag{48}$$

$$\sigma_1 = \tau_2(\rho_2 f_\alpha C_{h\alpha} + \eta_2 g_\alpha S_{h\alpha}), \qquad \sigma_2 = \tau_3(\rho_3 f_\beta C_\beta + \eta_3 g_\beta S_\beta)$$
$$\sigma_3 = \tau_1(\rho_1 f_\gamma C_\gamma + \eta_1 g_\gamma S_\gamma) \tag{49}$$

Note that $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$; k_α, k_β, k_γ; g_α, g_β, g_γ; f_α, f_β, f_γ; and $C_{h\alpha}$, C_β, C_γ, $S_{h\alpha}$, S_β, S_γ are already defined in Eqs. (20), (22), (30), (31), and (35), respectively, but must be calculated for the particular natural frequency ω_n at which the mode shape is required.

Thus, the mode shape of the axially loaded composite beam is given in explicit form by rewriting Eqs. (16–18) with the help of Eqs. (19–22) in the form

$$H(\xi) = A_1(\cosh\alpha\xi + R_1\sinh\alpha\xi + R_2\cos\beta\xi + R_3\sin\beta\xi + R_4\cos\gamma\xi + R_5\sin\gamma\xi) \tag{50}$$

$$\Theta(\xi) = A_1(\bar{\alpha}\sinh\alpha\xi + R_1\bar{\alpha}\cosh\alpha\xi - R_2\bar{\beta}\sin\beta\xi + R_3\bar{\beta}\cos\beta\xi - R_4\bar{\gamma}\sin\gamma\xi + R_5\bar{\gamma}\cos\gamma\xi)/L \tag{51}$$

$$\Psi(\xi) = A_1(k_\alpha\sinh\alpha\xi + R_1 k_\alpha\cosh\alpha\xi - R_2 k_\beta\sin\beta\xi + R_3 k_\beta\cos\beta\xi - R_4 k_\gamma\sin\gamma\xi + R_5 k_\gamma\cos\gamma\xi)/L \tag{52}$$

where the ratios R_1, R_2, R_3, R_4, and R_5 are A_2/A_1, A_3/A_1, A_4/A_1, A_5/A_1 and A_6/A_1, respectively, and follow from Eqs. (39).

Summary

The expressions for the frequency equation and mode shapes given by Eq. (34) and Eqs. (50–52) can now be used to compute the natural frequencies and mode shapes of axially loaded composite Timoshenko beams. This is demonstrated by an example taken form the literature.[2] It is a cross ply of AS/3501 carbon–epoxy laminated beam with stacking sequency [0/90/90/0 deg]. With the material properties given in Ref. 2, the rigidity properties and other beam parameters were calculated as follows: $EI = 1.070 \times 10^6$ Nm^2, $GJ = 1.380 \times 10^5$ Nm^2, $K = 0.001518$ Nm^2, $kAG = 3.163 \times 10^7$ N, $m = 13.892$ kg/m, and $I_\alpha = 0.02315$ kgm. Note that the cross-ply stacking sequence of the composite beam resulted in a small value of the bending–torsion coupling rigidity K as expected. The width-to-thickness ratio is taken to be unity, and the length-to-width ratio is taken to be 10, to make the results directly comparable with those of Ref. 2. The axial load P is taken to be 0 and 1.016×10^7 N (compressive), which correspond to $p^2 = 0$ and 0.95, respectively [see Eq. (12)].

The first five nondimensional natural frequencies ω_i^*, $i = 1, 2, \ldots, 5$, where $\omega_i^* = \omega L^2(\rho/E_1 h^2)^{1/2}$, and mode shapes of the cantilever beam obtained using the present theory, and with and without the effects of shear deformation, rotatory inertia, and axial load are shown in Fig. 2. The corresponding natural frequencies reported in Ref. 2 are shown in parentheses of Fig. 2b; note that Timoshenko beam theory is used in Ref. 2, but it does not account for the effect axial load and also does not predict the torsional frequencies. (NA in Fig. 2b stands for Not Available.) The results show that the shear deformation, rotatory inertia, and axial load have significant effects on the flexural frequencies of the beam. The agreement between the results using the present theory with those of Ref. 2 is very good, as can be seen. An interesting feature of the results is that when

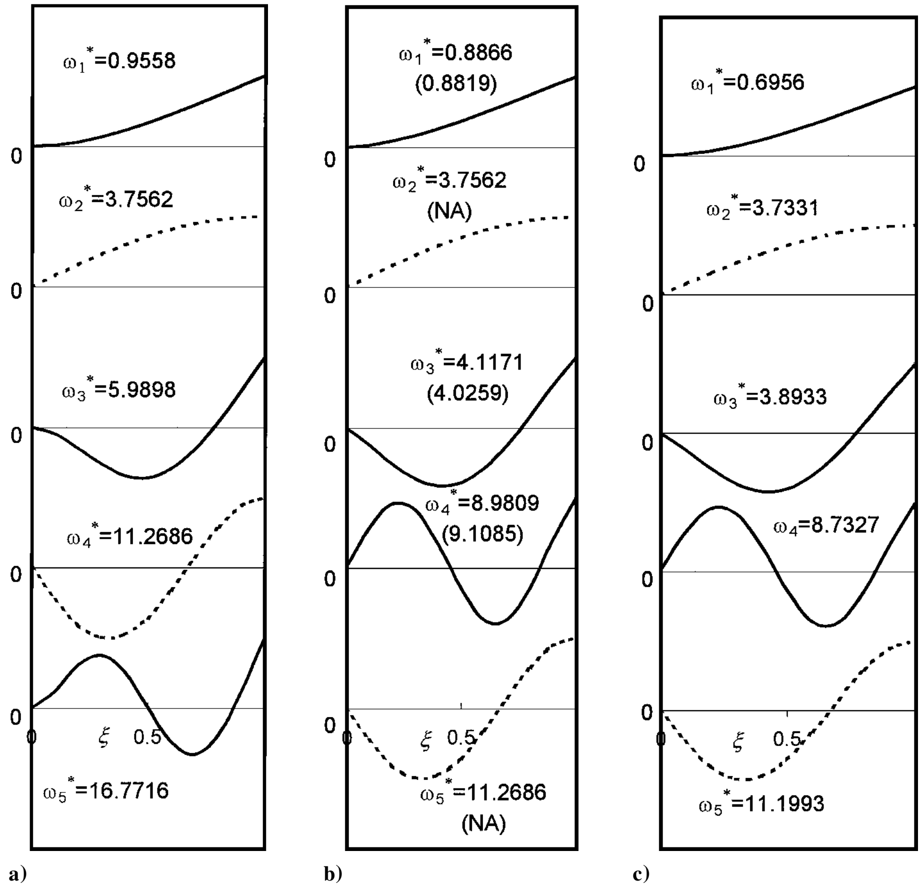

Fig. 2 Effects of shear deformation, rotatory inertia, and axial force on the natural frequencies and mode shapes of a composite beam,[2] with ——, bending displacement H and - - - -, torsional rotation Ψ: a) $p^2 = 0$, $r^2 = 0$, and $s^2 = 0$; b) $p^2 = 0$, $r^2 = 0.00083333$, and $s^2 = 0.033833$; and c) $p^2 = 0.95$, $r^2 = 0.00083333$, and $s^2 = 0.033833$.

the shear deformation, rotatory inertia, and axial load are ignored, the fourth normal mode of the beam is torsional, whereas, when the effects are included, this mode is shifted to the fifth position. The interchange (or flip over) between modes as a consequence of including the effects of shear deformation, rotatory inertia, and/or ply angle in a composite beam is significant from an aeroelastic point of view. The results show that the axial load changes the flexural frequencies, but has virtually no effect on torsional frequencies as expected. Because of the very small amount of coupling present in the cross-ply laminate, the modes generated are effectively uncoupled showing either bending or torsional displacements.

References

[1]Hodges, D. H., Atilgan, A. R., Fulton, M. V., and Rehfield, L. W., "Free Vibration Analysis of Composite Beams," *Journal of the American Helicopter Society*, Vol. 36, No. 3, 1991, pp. 36–47.

[2]Abramovich, H., and Livshits, A., "Free Vibrations of Non-symmetric Cross-ply Laminated Composite Beams," *Journal of Sound and Vibration*, Vol. 176, No. 5, 1994, pp. 597–612.

[3]Banerjee, J. R., and Williams, F. W., "Free Vibration of Composite Beams: An Exact Method Using Symbolic Computation," *Journal of Aircraft*, Vol. 32, No. 3, 1995, pp. 636–642.

[4]Banerjee, J. R., "Free Vibration of Axially Loaded Composite Timoshenko Beams Using the Dynamic Stiffness Method," *Computers and Structures*, Vol. 69, No. 2, 1998, pp. 197–208.

[5]Shirk, M. H., Hertz, T. J., and Weisshaar, T. A., "Aeroelastic Tailoring-Theory, Practice and Promise," *Journal of Aircraft*, Vol. 23, No. 1, 1986, pp. 6–18.

[6]Beltzer, A. I., "Engineering Analysis via Symbolic Computation—A Break Through," *Applied Mechanics Reviews*, Vol. 43, No. 6, 1990, pp. 119–127.

[7]Banerjee, J. R., "Explicit Analytical Expressions for Frequency Equation and Mode Shapes of Composite Beams," *International Journal of Solids Structures*, Vol. 38, No. 14, 2001, pp. 2415–2426.

[8]Banerjee, J. R., "Frequency Equation and Mode Shape Formulae for Composite Timoshenko Beams," *Composite Structures*, Vol. 51, No. 4, 2001, pp. 381–388.

[9]Hearn, A. C., "REDUCE User's Manual," Ver. 3.5. RAND, Santa Monica, CA, 1993.